LECTURE 12 THE CHAIN RULE

A horse is carrying a carriage on a dirt path. The amount of (energy) E (in calories) expended by the horse depends on the (distance) m (in miles) the horse walks. Also, the distance walked by the horse depends on (time) t (in hours). If the horse expends 40 calories per mile and the horse walks at a speed of 8 mph, at what rate is the horse expending energy (calories/hour)?

Intuitively, one would multiply these two rates to get the rate of energy expenditure. How is it related to the Chain Rule?

Consider the energy per mile as $\frac{dE}{dm}$ and the distance per hour as $\frac{dm}{dt}$. We want $\frac{dE}{dt}$, the rate of energy expenditure, with units in calories per hour, $\frac{dE}{dt}$. It seems we are doing the following,

$$\frac{dE}{dt} = \frac{dE}{dm} \cdot \frac{dm}{dt}.$$

If you do dimensional analysis, this all makes sense.

In more proper mathematical terms, the energy of the horse is express as $E\left(m\left(t\right)\right)$, a composed function. Therefore,

$$\frac{d}{dt}\left(E\left(m\left(t\right)\right)\right) = \frac{d}{dm}\left(E\left(m\right)\right) \cdot \frac{d}{dt}\left(m\left(t\right)\right) = E'\left(m\left(t\right)\right) \cdot m'\left(t\right).$$

In practice, it is useful to observe which function is the outer function, E, and which one is the inner one, m. It takes practice to realise what's happening.

Example 1. Find the function composition $f \circ u$ with the following components.

(1) $f\left(x\right) = x^3$, $u\left(x\right) = e^{5x}$.
(2) $f\left(x\right) = \sin\left(x\right)$, $u\left(x\right) = 5x - 1$.
(3) $f\left(x\right) = \sqrt{x}$, $u\left(x\right) = \sin^2\left(x\right)$.

Now, we must know how to go backwards in the above process, that is, we must identify the components from a given function.

Remark. It is always useful/instructive to use u as the independent variable for the outer function f to realise it is $f\left(u\right)$, and then use x for $u\left(x\right)$, so that the structure of $f \circ u = f\left(u\left(x\right)\right)$ is clear. When you are given a function, you think about how you would numerically evaluate it. The operation you do first is the inner most function, while the last operation is the outer most function.

Example 2. Identify the outer and inner function of the following composite functions. Then, find their derivatives.

(1) $f\left(x\right) = \left(3x^2 + 1\right)^2$.

Solution. Outer function is $f\left(u\right) = u^2$ and $u\left(x\right) = 3x^2 + 1$. Why? To evaluate $f\left(x\right)$ in a calculator for a particular x, you would first find out what $3x^2 + 1$ is first, then square it. So the inner function is $u\left(x\right) = 3x^2 + 1$. Then, you square it, with outer function $f\left(u\right) = u^2$.

To confirm, you take a composition,

$$\left(f \circ u\right)\left(x\right) = f\left(u\left(x\right)\right) = u\left(x\right)^2 = \left(3x^2 + 1\right)^2.$$

$$\frac{d}{dx} f\left(u\left(x\right)\right) = \frac{df}{du} \cdot \frac{du}{dx} = 2u \mid_{u=3x^2+1} \cdot \left(6x\right) = 2\left(3x^2 + 1\right) \cdot 6x = 12x\left(3x^2 + 1\right).$$

or

$$\frac{d}{dx} f\left(u\left(x\right)\right) = f'\left(u\left(x\right)\right) u'\left(x\right) = 2\left(3x^2 + 1\right) \cdot 6x = 12x\left(3x^2 + 1\right).$$

(2) $g\left(z\right) = \left(z + 1\right)^{-3}$.

Solution. Outer function is $g(u) = u^{-3}$ and $u(z) = z + 1$. To confirm, we find

$$g \circ u = g(u(z)) = u^{-3}(z) = (z+1)^{-3}.$$

Then,

$$\frac{d}{dz} g(u(z)) = \frac{dg}{du} \cdot \frac{du}{dz} = -3u^{-4}\mid_{u=z+1} \cdot (1) = -3(z+1)^{-4}.$$

Example 3. Suppose the bacteria population evolves according to the following function

$$y(t) = 1000e^{5t} - 300$$

where t represents the number of days. Find the rate at which the bacteria is growing on the fifth day.

Solution. Since we are talking about rate of growth, we need to find the derivative of the population with respect to time. The outer function here is

$$y(u) = 1000e^{u} - 300$$

and inner function is

$$u(t) = 5t.$$

To confirm,

$$y \circ u(t) = y(u(t)) = 1000e^{u(t)} - 300 = 1000e^{5t} - 300$$

check!

$$y'(t) = \frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = 1000e^{u}\mid_{u=5t} \cdot (5) = 5000e^{5t}.$$

At the fifth day, we find the rate by evaluating

$$y'(5) = 5000e^{5\times 5} = 5000e^{25}$$

(a pretty big number).

Example 4. More patterns for the derivative of functions of certain forms.

We know that $\frac{d}{dx}e^{x} = e^{x}$. Can we find a nice pattern for functions composed with the exponential functions, $e^{u(x)}$?So, suppose $f(x) = e^{\cos(x)}$. Then, we identify outer function $f(u) = e^{u}$ and inner function $u(x) = \cos(x)$.

$$\frac{d}{dx} f(u(x)) = \frac{df}{du} \cdot \frac{du}{dx} = e^{u}\mid_{u=\cos(x)} \cdot (-\sin(x)) = -\sin(x)\, e^{\cos(x)}$$

where we observe we always retain the original function $f(x)$ as part of the product.

Thus, we propose the following pattern for $f(x) = e^{u(x)}$

$$\frac{d}{dx} e^{u(x)} = \frac{d}{du} e^{u} \times \frac{d}{dx} u(x) = e^{u(x)} u'(x) = f(x)\, u'(x).$$